properties of inner products. Then we turn to the task of discussing length and orthogonality.

Definition. *Let* F *be the field of real numbers or the field of complex numbers, and* V *a vector space over* F. *An* **inner product** *on* V *is a function which assigns to each ordered pair of vectors* α, β *in* V *a scalar* $(\alpha|\beta)$ *in* F *in such a way that for all* α, β, γ *in* V *and all scalars* c

(a) $(\alpha + \beta|\gamma) = (\alpha|\gamma) + (\beta|\gamma)$;

(b) $(c\alpha|\beta) = c(\alpha|\beta)$;

(c) $(\beta|\alpha) = \overline{(\alpha|\beta)}$, *the bar denoting complex conjugation*;

(d) $(\alpha|\alpha) > 0$ *if* $\alpha \neq 0$.

It should be observed that conditions (a), (b), and (c) imply that

(e) $$(\alpha|c\beta + \gamma) = \bar{c}(\alpha|\beta) + (\alpha|\gamma).$$

One other point should be made. When F is the field R of real numbers, the complex conjugates appearing in (c) and (e) are superfluous; however, in the complex case they are necessary for the consistency of the conditions. Without these complex conjugates, we would have the contradiction:

$$(\alpha|\alpha) > 0 \quad \text{and} \quad (i\alpha|i\alpha) = -1(\alpha|\alpha) > 0.$$

In the examples that follow and throughout the chapter, F is either the field of real numbers or the field of complex numbers.

EXAMPLE 1. On F^n there is an inner product which we call the **standard inner product.** It is defined on $\alpha = (x_1, \ldots, x_n)$ and $\beta = (y_1, \ldots, y_n)$ by

(8-1) $$(\alpha|\beta) = \sum_j x_j\bar{y}_j.$$

When $F = R$, this may also be written

$$(\alpha|\beta) = \sum_j x_jy_j.$$

In the real case, the standard inner product is often called the dot or scalar product and denoted by $\alpha \cdot \beta$.

EXAMPLE 2. For $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ in R^2, let

$$(\alpha|\beta) = x_1y_1 - x_2y_1 - x_1y_2 + 4x_2y_2.$$

Since $(\alpha|\alpha) = (x_1 - x_2)^2 + 3x_2^2$, it follows that $(\alpha|\alpha) > 0$ if $\alpha \neq 0$. Conditions (a), (b), and (c) of the definition are easily verified.

EXAMPLE 3. Let V be $F^{n\times n}$, the space of all $n \times n$ matrices over F. Then V is isomorphic to F^{n^2} in a natural way. It therefore follows from Example 1 that the equation

$$(A|B) = \sum_{j,k} A_{jk}\overline{B}_{jk}$$